

THEORY OF REPRODUCING KERNELS FOR HILBERT SPACES
OF VECTOR VALUED FUNCTIONS

by

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Introduction

The general theory of reproducing kernels developed by N. Aronszajn in [1] and [2] provides a unifying point of view for the study of an important class of Hilbert spaces of real or complex valued functions and for the application of the methods of Hilbert space theory to different problems in the theory of partial differential equations.

With a view to applications to systems of such equations the form which the theory takes in the case of spaces of vector valued functions was investigated, initially for finite dimensional and Hilbert range spaces. It was found that the natural setting for such a generalization of the theory is that in which the functions of the functional Hilbert space take their values in an arbitrary locally convex linear topological space, since all of the main results are essentially preserved in that setting and a more special case would restrict unduly the applications.

The present study is confined to the exposition of the general theory with a few illustrations and undertakes to extend the basic notions of proper functional space, reproducing kernel and positive matrix and their properties as they occur in the paper of N. Aronszajn [1] (referred to as the usual theory). These notions are introduced in the first two sections where there is also presented a device whereby problems in the present theory can be reduced to problems in the usual theory and their solution effected by applying the existing theorems and seeking the proper form of the result in the general setting. In many cases this offers no difficulty and it only remains to find the implications of the result which are peculiar to the present setting. Such is the case for restrictions, sums and differences and related

questions considered in Sections 4 and 6. The third section gives results on the effect on the kernel of continuous transformations of the range space.

The cases in which the usual theory is not immediately translated are dependent upon the central question of the existence of functional completions and this problem is discussed in Section 5. The last two sections take up results of this nature. The notion of product in our setting requires the introduction of the tensor product of locally convex spaces (see N. Bourbaki [5], J. Dieudonné [7] and A. Grothendieck [8]) and linear transformations. Finally, using the general notions of inductive and projective systems the limits of reproducing kernels of our type are studied.

The main source of information on locally convex spaces in doing this work was a draft version of a book written by a group of mathematicians at the University of Kansas in 1953 and soon to be published in final form, see J. L. Kelley [9]. The most readily available general reference on the subject is the book of N. Bourbaki [6], (see also J. Dieudonné [7]).

We have summarized the results which we need in the preliminary section. This was done largely because we have preferred to deal with anti-linear functionals rather than linear functionals as is usual in the theory of locally convex spaces and our notations are therefore not standard ones. This choice was motivated by the Hilbert space background of our subject.

The necessary information from the general theory of Hilbert spaces is to be found in N. Aronszajn [3] and M. H. Stone [10].

Preliminaries.

This section is devoted to a summary of certain well known definitions and results, mainly from the theory of linear topological spaces, and is intended to serve as a means of establishing notations and conventions in the forms which are most convenient for our later purposes.

If U denotes a linear space then the algebraic dual U' of U is taken to be the space of all antilinear functionals on U . An antilinear functional on U is a scalar (i.e., real or complex) valued function u' on U such that, if $\langle u', u \rangle$ denotes the value of u' at $u \in U$,

$$\langle u', \alpha u_1 + \beta u_2 \rangle = \bar{\alpha} \langle u', u_1 \rangle + \bar{\beta} \langle u', u_2 \rangle ,$$

for all scalars α, β and all $u_1, u_2 \in U$.

The operations of addition and scalar multiplication in U' are defined pointwise on U so that

$$\langle \alpha u'_1 + \beta u'_2, u \rangle = \alpha \langle u'_1, u \rangle + \beta \langle u'_2, u \rangle .$$

In short, $\langle u', u \rangle$ is linear in the first variable and antilinear in the second. This usage departs from the usual convention in linear topological space theory but conforms to that of Hilbert space theory where the scalar product $(\ , \)$ has the above property, i.e. is Hermitian bilinear. The notations $(\ , \)$ and $\| \ \|$ are used for the scalar product and norm in all Hilbert spaces and are provided with subscripts identifying the space only when necessary to avoid confusion.

If U'' denotes the algebraic dual of U' then for any $u \in U$, $\langle u', u \rangle$ is a linear functional on U' and hence $\overline{\langle u', u \rangle}$ is a member of U'' . The correspondence thus established between U and a subspace of U'' is linear

and one-one, i.e. is an isomorphism giving a canonical identification of U with a subspace of U'' .

U becomes a linear topological space if we define there a topology such that the operations of addition and scalar multiplication are continuous. If $u \in U$ then any neighborhood of u is of the form $u+N$ where N is a neighborhood of the point zero. Thus the topology for U is completely specified by giving the system of neighborhoods of zero. If there is a basis for the system of neighborhoods of zero consisting of convex sets then U is a locally convex space. We shall consider Hausdorff spaces almost exclusively and adopt the abbreviation "l. c. s." for Hausdorff locally convex space.

The subspace of U' consisting of all functionals which are continuous in the topology of an arbitrary locally convex U is denoted U^* and called the conjugate of U and a necessary and sufficient condition for U to be Hausdorff is that U^* distinguish elements of U , i.e. for every non-zero $u \in U$, there exists a $u^* \in U^*$ such that $\langle u^*, u \rangle \neq 0$.

Every subspace U^* of U' defines a topology on U , the linear¹ topology, denoted $w(U, U^*)$, defined to be the weakest topology on U in which the functionals of U^* are continuous.

The neighborhoods of zero in $w(U, U^*)$ are defined as the sets consisting of all $u \in U$ such that

$$|\langle u_i^*, u \rangle| < \epsilon, \quad i = 1, 2, \dots, n$$

for any $\epsilon > 0$ and $u_1^*, u_2^*, \dots, u_n^* \in U^*$.

1. The term weak topology has been used widely for this notion but we adopt the term linear topology as being more suggestive and in order to avoid such phrases as weakest weak and strongest weak in comparing topologies.

Clearly U^* is the conjugate of U under the $w(U, U^*)$ topology.

The linear topology $w(U', U)$ induced in U' by the subspace U of U'' is called the weak-star topology. It is characterized as the topology of pointwise convergence of elements of U' at each point of U . An antilinear functional on U' is continuous in the weak-star topology if and only if it is the image of an element $u \in U$ under the natural identification. In particular, the conjugate of U' under $w(U', U)$ is U . If U'_0 is a subspace of U' we denote the relativised weak-star topology on U'_0 by $w(U'_0, U)$. U'_0 is weak-star dense in U' if and only if it distinguishes elements of U .

If U is a l.c.s. and U^* is its conjugate space under the given topology then $w(U, U^*)$ is the weakest topology for U such that U^* is the conjugate space. There is also a strongest such topology, the Mackey topology $m(U, U^*)$.

The neighborhoods of zero in $m(U, U^*)$ are defined as follows. For every $\epsilon > 0$ and every convex weak-star compact subset C of U^* , the set of all $u \in U$ such that $|\langle u^*, u \rangle| < \epsilon$ whenever $u^* \in C$ is a neighborhood of zero.

When U is a normed linear space $m(U, U^*)$ coincides with the norm topology. When U is a Hilbert space the conjugate space is identified with U and the topologies $w(U, U^*)$ and $w(U^*, U)$ coincide and give the usual weak topology of the Hilbert space.

For our purposes the space U^* is generally considered with the weak-star topology so that its conjugate is U and we have no need to speak of U^{**} .

A stronger topology on U^* than $w(U, U^*)$ determines a subspace of $(U^*)'$ which contains U and conversely any such subspace determines on U^* the stronger linear topology which it induces there. We shall call any space containing U which is thus obtained by strengthening the topology of U^* , a completion of U . Any completion of U can be topologised so that the relative topology in U is its given topology. This follows from the fact that the relativised Mackey topology of the completion is stronger than $m(U, U^*)$ and hence the completion can be taken under the topology consisting of those open sets in its Mackey topology which intersect U in an open set in the given topology of U . U is dense in any of its completions since otherwise there would be a continuous linear functional on the completion vanishing on the closure of U but not vanishing identically and therefore the conjugate spaces would be different, which is impossible.

A l. c. s. is in particular a uniform structure and as such there is a notion of completion resulting from the general theory of uniform structures. However, we shall make use of completion with respect to the given topology only in case of normed spaces where the usual construction by classes of equivalence of Cauchy sequences suffices.

A space is sequentially complete if it contains a limit element for each of its Cauchy sequences.

If T is a linear transformation of a l. c. s. U into a l. c. s. V then the adjoint of T , when it exists, is a linear transformation T^* of V^* into U^* such that

$$\langle v^*, Tu \rangle = \langle T^* v^*, u \rangle ,$$

for all $u \in U$, $v^* \in V^*$. If T^* exists it is unique. T^* exists if and only if

T is continuous relative to the linear topologies $w(U, U^*)$ and $w(V, V^*)$. When T^* exists it is continuous in the weak-star topologies $w(V^*, V)$ and $w(U^*, U)$.

If T is continuous in the linear topologies induced by subspaces U_1^* and V_1^* of U' and V' then it is continuous in the Mackey topologies $m(U, U_1^*)$ and $m(V, V_1^*)$ and conversely. The converse follows from the general fact that continuity in the given topologies implies continuity in the linear topologies which they determine. When T has the above equivalent continuity properties we shall say that T is $U_1^* - V_1^*$ continuous. T is continuous, simply, will mean that T is continuous in the given topologies of U and V . In summary, when T is continuous it is $U^* - V^*$ continuous and this is equivalent to the existence of T^* , which is then $V - U$ continuous.

When the domain U is a Hilbert space we may omit the U^* and speak of a V^* -continuous transformation of U into V , and a V -continuous transformation of V^* into U . The former is continuous in the strong topology of U together with any topology from $w(V, V^*)$ to $m(V, V^*)$ as well as in the weak topology of U and the linear topology of V . The latter includes continuity in the weak-star topology of V^* and the weak topology of U , as well as the $m(V^*, V)$ -strong continuity.

In the sequel we shall be especially interested in the case where V is a given l.c.s. R and U is R^* . Here, the continuity of the transformation T of R^* into R in the $R - R^*$ sense implies the existence of T^* (and conversely) and T^* is also a transformation of R^* into R (by means of the canonical identification) with the same continuity property.

We shall call such a transformation T self adjoint when $T = T^*$. An

equivalent formulation of the condition is

$$\langle r^*, Ts^* \rangle = \overline{\langle s^*, T^*r^* \rangle} = \overline{\langle s^*, Tr^* \rangle}$$

for all $r^*, s^* \in R^*$, i. e. the Hermitian symmetry of the Hermitian bilinear form $\langle r^*, Ts^* \rangle$ on R^* . This is in turn equivalent to the reality of the quadratic form $\langle r^*, Tr^* \rangle$. We shall say that T is positive if

$$\langle r^*, Tr^* \rangle \geq 0 \quad \text{for all } r^* \in R^*.$$

If U_0 is a subspace of a l. c. s U then the polar of U_0 , U_0^\perp , is the subspace of U^* consisting of all functionals there which vanish on U_0 . U_0^\perp is closed in the $w(U^*, U)$ topology. The definition applies also to a subspace of U^* , giving as the polar a $w(U, U^*)$ closed subspace of U .

The nullspace of a linear transformation is the subspace of its domain consisting of all elements which are transformed into zero. If T is a continuous linear transformation of U into V then T^* exists and the nullspaces \mathcal{N}_T and \mathcal{N}_{T^*} and the ranges R_T and R_{T^*} are related by

$$R_T^\perp = \mathcal{N}_{T^*} \quad \text{and} \quad R_{T^*}^\perp = \mathcal{N}_T.$$

If the linear space U is a subspace of the linear space V then the quotient space V/U of $V \bmod U$ is the linear space whose elements are classes of equivalence of members of V under the relation: v_1 is equivalent to v_2 if $v_1 - v_2 \in U$. The member of V/U containing v is the set $v+U$ and the mapping of V onto V/U which carries v into $v+U$ is called the quotient map. If V is a l. c. s. then V/U is topologised by defining a set there to be open if and only if its inverse image under the quotient map is open in V . The space V/U is then locally convex and is Hausdorff if and only if U is a closed subspace of V . The quotient map

is continuous and open when V/U has the above quotient topology derived from the given topology of V .

If V has a topology in which it is not a Hausdorff space, the closed subspace generated by taking the closure of the set whose only member is zero will be called the zero subspace of the topology. V is made into a Hausdorff space by forming its quotient space mod. the zero subspace.

Letting Q stand for the quotient map of V onto V/U , Q^* exists and for $w \in (V/U)^*$, $\langle Q^*w, v \rangle = \langle w, v+U \rangle$. Q is continuous relative to the linear topologies (and the Mackey topologies) and Q^* is continuous relative to the weak-star topologies (and the Mackey topologies) of the conjugate spaces. Finally, if U is closed then Q^* is a topological isomorphism of $(V/U)^*$ onto U^\perp with the weak-star topologies.

If U is a subspace of V then the canonical mapping which carries $u \in U$ into u as a member of V is called the injection map I . Its adjoint is the mapping which assigns to $v^* \in V^*$ its restriction to U . The nullspace of I^* is $U^\perp \subset V^*$ and if U is closed in V and Q is the quotient map of V^* onto V^*/U^\perp then I^*Q^{-1} is a topological isomorphism of V^*/U^\perp onto U^* .

We shall need the algebraic tensor product of two linear spaces R and S . This is denoted $R \otimes S$ and is defined by the following procedure. For every pair (r, s) of elements of R and S we have a well determined function on $R' \times S'$, the Cartesian product of the duals, namely the product $\langle r', r \rangle \langle s', s \rangle$. Distinct pairs may give the same function. If we define two pairs to be equivalent when they yield the same function then $R \times S$ is divided into classes of equivalence by this relation. We may denote by $r \otimes s$ either the class of equivalence of (r, s) or the function which it

determines. The space $R \otimes S$ is the linear space generated by these elementary products in the space of all functions on $R' \times S'$ with scalar values, i.e. the space of all finite linear combinations $\sum_{i=1}^n a_i r_i \otimes s_i$.

The representation of elements of $R \otimes S$ by such sums is not unique and in fact, since $a \cdot r \otimes s = (a r) \otimes s = r \otimes (a s)$, the general element of $R \otimes S$ can be written $\sum_{i=1}^n r_i \otimes s_i$.

The basic property of the space $R \otimes S$ is that its linear functionals can be identified with the bilinear functionals on $R \times S$. A bilinear functional (to be distinguished from "Hermitian bilinear functional") is a complex valued function $B(r, s)$ which is linear in each of its variables for every fixed value of the other.

The dual space of $R \otimes S$ is then identified as follows. If L is any antilinear functional on $R \otimes S$ then the conjugate of its restriction to the elementary products is a bilinear functional on $R \times S$,

$$\overline{L(r \otimes s)} = B(r, s) .$$

Conversely, any bilinear functional on $R \times S$ determines by this formula the values of an antilinear functional L on the elementary products and hence, by antilinear extension, a unique element of $(R \otimes S)'$.

If each of the spaces R and S is a l.c.s. and R^* and S^* are their conjugates, then, since $\overline{R^*}$ and $\overline{S^*}$ are weak-star dense in R' and S' , functions of the form $\overline{\langle r', r \rangle} = \overline{\langle s', s \rangle}$ are identified with those of the form $\overline{\langle r^*, r \rangle} = \overline{\langle s^*, s \rangle}$ and therefore $R \otimes S$ is a subspace of the algebraic dual of $R^* \otimes S^*$. Each of these spaces acquires a linear topology in this way and becomes a l.c.s. .

We denote by $R \hat{\otimes}_\tau S$ the completion of $R \otimes S$ defined by a topology τ in $R^* \hat{\otimes} S^*$ which is stronger than the linear topology there.

If R and S are Hilbert spaces then $R \hat{\otimes} S$ and $R^* \hat{\otimes} S^*$ are identified and the resulting space, $R \hat{\otimes} S$, has a well determined norm given by

$$\left\| \sum_i r_i \hat{\otimes} s_i \right\|^2 = \sum_{i,j} (r_i, r_j)(s_i, s_j) .$$

With this topology as τ we obtain a completion of $R \hat{\otimes} S$ which is a Hilbert space and which we denote by $R \hat{\otimes}_h S$, the Hilbert space tensor product.

If T_1 and T_2 are linear transformations of R_1 into R and S_1 into S respectively, the tensor product $T_1 \hat{\otimes} T_2$ is the linear transformation of $R_1 \hat{\otimes} S_1$ into $R \hat{\otimes} S$ defined on the elementary products by

$$(T_1 \hat{\otimes} T_2)(r_1 \hat{\otimes} s_1) = T_1 r_1 \hat{\otimes} T_2 s_1$$

and extended by linearity.

If $\{V_n\}$ is a sequence of linear spaces and we form the Cartesian product consisting of all sequences $\{v_n\}$ such that $v_n \in V_n$ for each n , then a structure of a linear space is defined there by defining the addition and scalar multiplication coordinatewise, i.e. $\alpha \{v_n^{(1)}\} + \beta \{v_n^{(2)}\} = \{\alpha v_n^{(1)} + \beta v_n^{(2)}\}$. The resulting linear space is called the product $\prod_{n=1}^{\infty} V_n$ of the spaces V_n . For each n , the linear transformation of the product into V_n which assigns to an element of the product its n -th coordinate is called the projection on V_n .

If each V_n is a l.c.s. then the product is a l.c.s. under the product topology defined as the weakest topology such that all of the projections are

continuous.

The subspace of the product of the linear spaces V_n consisting of all $\{v_n\}$ with only a finite number of non-zero coordinates is called the direct sum of the spaces V_n , denoted $\sum_{n=1}^{\infty} V_n$. The projections of the product onto its separate coordinate spaces induce projections of the direct sum onto its terms.

If V'_n are the algebraic duals of the linear spaces V_n then $\prod_{n=1}^{\infty} V'_n$ is the algebraic dual of $\sum_{n=1}^{\infty} V_n$, where, for $\{v'_n\} \in \prod_{n=1}^{\infty} V'_n$ and $\{v_n\} \in \sum_{n=1}^{\infty} V_n$ we put

$$\langle \{v'_n\}, \{v_n\} \rangle = \sum \langle v'_n, v_n \rangle$$

the sum being extended over the finite set of indices for which $v_n \neq 0$.

If each V_n is a l. c. s. we topologize $\sum V_n$ with the strongest topology which induces in each term V_n , considered as subspace of the sum, its given topology. With this definition the direct sum is a l. c. s. . If V_n^* is conjugate of V_n then the conjugate of $\sum_{n=1}^{\infty} V_n$ is $\prod_{n=1}^{\infty} V_n^*$, with the values of elements of $\prod V_n^*$ as functionals on $\sum V_n$ given by the above formula. In the same way the conjugate of $\prod V_n$ is $\sum V_n^*$.

The direct sum of two spaces is written $U \dot{+} V$ and since we may identify $u \in U$ with $(u, 0) \in U \dot{+} V$ and similarly for $v \in V$, the fact that $(u, v) = (u, 0) + (0, v)$ allows us to denote the elements of the sum by $u+v$ instead of (u, v) .

For a finite number of spaces the direct sum and the product are the

same. In particular $(U \dot{+} V)^*$ and $U^* \dot{+} V^*$ are identified, i.e., by

$$\langle u^* + v^*, u + v \rangle = \langle u^*, u \rangle + \langle v^*, v \rangle .$$

A system $\{E_n, \pi_n\}$ consisting of a sequence of sets E_n and mappings π_n is said to be: a) an inductive system if for each $n = 1, 2, \dots$, π_n is a mapping of E_n into E_{n+1} ; b) a projective system if for each $n = 1, 2, \dots$, π_n is a mapping of E_{n+1} into E_n .

These notions are extensions of the notions of increasing and decreasing sequences of sets respectively, in which case the π_n are just the identity mappings.

As extensions of the union and intersection of such sequences we introduce the limits of inductive and projective systems as follows.

The inductive limit, $I\text{-}\lim\{E_n, \pi_n\}$ of an inductive system is obtained from the set of all sequences $\{e_n\}$ where $e_n \in E_n$ and $n = n_0, n_0+1, \dots$, for some n_0 , which have the property that $\pi_k e_k = e_{k+1}$ for $k \geq n_0$, by taking classes of equivalence according to the relation $\{e_n\}$ equivalent to $\{e'_n\}$ if for some n , $e_n = e'_n$ (the sequences are then identical for all larger n).

The projective limit, $P\text{-}\lim\{E_n, \pi_n\}$, of a projective system is the set of all sequences $\{e_n\}$, $n = 1, 2, \dots$, such that $e_n \in E_n$ and $e_n = \pi_{n+1} e_{n+1}$ for all n .

There is a well determined mapping of each E_n in an inductive system into $I\text{-}\lim\{E_n, \pi_n\}$ given by $\pi_{n\infty}$ where

$$\pi_{n\infty} e_n = \{e_n, \pi_n e_n, \pi_{n+1} \pi_n e_n, \dots\} .$$

For these mappings we have the property $\pi_{n\infty} = \pi_{(n+1)\infty} \pi_n$.

In the projective case we have for each n a mapping of $P\text{-lim}\{E_n, \pi_n\}$ into E_n given by $\pi_{n\infty}\{e_i = e_n\}$.

Let $\{f_n\}$ be a sequence of functions with f_n defined on E_n for each n .

a) If $\{E_n, \pi_n\}$ is an inductive system then we shall call $\{f_n\}$ an inductive sequence if for all n

$$f_{n+1}(\pi_n e_n) = f_n(e_n) \quad \text{for all } e_n \in E_n,$$

and define the inductive limit of $\{f_n\}$, $f_0 = I\text{-lim}_n f_n$, to be the function defined on $I\text{-lim}\{E_n, \pi_n\}$ by $f_0(\{e_i\}) = f_n(e_n)$ for any choice of n for which e_n is defined. Here $\{e_i\}$ is any sequence of a class of equivalence belonging to E . For all n we have $f_n(e_n) = f_0(\pi_{n\infty} e_n)$ and hence the functions on $I\text{-lim}\{E_n, \pi_n\}$ are in one-one correspondence with the inductive sequences.

b) If $\{E_n, \pi_n\}$ is a projective system then f_n is a projective sequence if for some n_0 and for all $e_{n+1} \in E_{n+1}$, $n \geq n_0$,

$$f_{n+1}(e_{n+1}) = f_n(\pi_n e_{n+1}) \quad \text{for } n = n_0, n_0+1, \dots$$

The projective limit, $P\text{-lim } f_n$, is the function $f_0(\{e_i\})$ defined on $P\text{-lim}\{E_n, \pi_n\}$ by

$$f_0(\{e_i\}) = f_n(e_n) \quad \text{for any } n \geq n_0.$$

Since this is equivalent to saying that $f_0(\{e_i\}) = f_n(\pi_{n\infty}\{e_i\})$ for $n \geq n_0$, we see that the function f_n may be changed on the complement in E_n of the range of $\pi_{n\infty}$ and that an altered sequence of functions can be constructed in this way which is a projective sequence and has f_0 as its P-limit. Consequently, there may be in general an infinite number of projective sequences having a given function on $P\text{-lim}\{E_n, \pi_n\}$ as their P-limit.

The notion of an inductive sequence of functions in the case of an increasing sequence of sets $\{E_n\}$ gives a sequence of functions defined in the successive sets and such that for all n the restriction of f_{n+1} to E_n is f_n . If E is the union of the E_n then each $e \in E$ is contained in E_n for n sufficiently large and putting $f(e) = f_n(e)$ for any such n we obtain the I-limit of $\{f_n\}$.

For a decreasing sequence of sets we see that a projective sequence of functions consists of functions f_n defined on E_n and such that for some n_0 and $n \geq n_0$, f_{n+1} is the restriction of f_n to E_{n+1} . The P-limit is the common restriction of all f_n , $n \geq n_0$, to the intersection E .

Suppose now that $\{V_n, \sigma_n\}$ is a system consisting of linear spaces V_n and linear transformations σ_n . If $\{V_n, \sigma_n\}$ is an inductive system we interpret $V = I\text{-}\lim \{V_n, \sigma_n\}$ as a linear space by choosing arbitrary sequences $\{v_n\}$ and $\{v_n^{(1)}\}$ from any two classes of equivalence and defining the linear combination of these classes to be the class of equivalence of $\{\alpha v_n + \beta v_n^{(1)}\}$. The definition is clearly independent of the choice of representatives.

There is a way of defining V directly as a quotient space of the direct sum $\sum V_n$. The present definition is more straightforward and suffices for our purposes.

If, now, $\{V_n, \sigma_n\}$ is projective and $\{v_n\}$ and $\{v_n^{(1)}\} \in V = P\text{-}\lim \{V_n, \sigma_n\}$ then $\{\alpha v_n + \beta v_n^{(1)}\} \in V$ and V is given a linear structure in this way. It is clear that V is a subspace of the product $\prod V_n$.

If the spaces V_n are l.c. spaces and the transformations σ_n are continuous then from the conjugate spaces we obtain the system $\{V_n^*, \sigma_n^*\}$.

If $\{V_n, \sigma_n\}$ is inductive then $\{V_n^*, \sigma_n^*\}$ is projective and if $\{V_n, \sigma_n\}$ is projective $\{V_n^*, \sigma_n^*\}$ is inductive. We define the expression $\langle \{V_n^*\}, \{V_m\} \rangle$ in the following natural way, choosing $\{V_n, \sigma_n\}$ to be inductive in order to fix the notations. In this case $\{V_m\}$ is supposed to be any member of a class of equivalence in $V = I\text{-}\lim \{V_n, \sigma_n\}$. We have

$$\langle V_n^*, V_n \rangle = \langle \sigma_n^* V_{n+1}^*, V_n \rangle = \langle V_{n+1}^*, \sigma_n V_n \rangle = \langle V_{n+1}^*, V_{n+1} \rangle ,$$

for all sufficiently large n . We take the fixed value so determined for $\langle \{V_n^*\}, \{V_n\} \rangle$ and notice that it is independent of the choice of the representative in V .

The topologies on the member spaces of an inductive or projective system serve to define natural topologies on the limits by means of the transformations $\sigma_{n\infty}$. Namely, if $\{V_n, \sigma_n\}$ is an inductive system then $\sigma_{n\infty}$ is a transformation of V_n into $V = I\text{-}\lim \{V_n, \sigma_n\}$ for each n and the topology of V is taken to be the strongest topology which makes every $\sigma_{n\infty}$ continuous. In case $V = P\text{-}\lim \{V_n, \sigma_n\}$ for a projective system $\{V_n, \sigma_n\}$ the mapping $\sigma_{n\infty}$ is a transformation of V into V_n and we give V the weakest topology such that $\sigma_{n\infty}$ is continuous in the given topology of V_n for every n . This topology on the projective limit is just the topology of $\prod V_n$ relativised to V as a subspace there.

If the spaces V_n are members of an inductive system and V^* is the projective limit of $\{V_n^*, \sigma_n^*\}$ then the $w(V^*, V)$ topology on V^* is the same as the topology defined there by the mappings $\sigma_{n\infty}^*$ when V_n^* is considered with $w(V_n^*, V_n)$. The space V will not in general be a Hausdorff space in this case and will become a l.c.s. only when the quotient space is taken modulo the zero subspace.

When $\{V_n, \sigma_n\}$ is a projective system the $w(V^*, V)$ topology is again the topology determined by the $\sigma_{n\infty}$ and the weak star topologies of the V_n^* . In this case V will be a l. c. s.

We shall often find it convenient to employ a special device in writing formulas to stress which of the variables in an expression are to be taken as free variables. For example, if $f(x, y)$ is a function defined on the rectangle $0 \leq x \leq a, 0 \leq y \leq b$ then for the function on the interval $[0, a]$ obtained by fixing y we will write $f(\dot{x}, y)$ and say " $f(\dot{x}, y)$ is continuous as a function of x , for each y ". It is not our intention to attempt a completely rigorous distinction between free and bound variables but merely to accent certain aspects of some of our formulas.

Finally, in dealing with Hilbert spaces, which we take to be non-separable in general, a complete orthonormal system will be indexed by α or β , e. g. $\{\varphi_\alpha\}$, and expansions such as $\sum_{\alpha} a_{\alpha} \varphi_{\alpha}$ are meant in the sense that all but an at most countable number of terms vanish and the remainder form a convergent series.

§ 1. Definition and basic properties of reproducing kernels.

Let E denote an abstract set, R a l.c.s. and F a Hilbert space of functions on E with values in R .

Definition. A reproducing kernel (r.k.) for F is a family $K(x, y)$ of transformations of R^* into R defined for each $x, y \in E$ and having the properties

- 1) for every $y \in E$ and $r^* \in R^*$, $K(\dot{x}, y)r^* \in F$,
- 2) for every $y \in E$, $f \in F$ and $r^* \in R^*$, $\langle r^*, f(y) \rangle = (K(\dot{x}, y)r^*, f(\dot{x}))$.

It is immediately seen that if a r.k. for F exists it is unique. The condition for the existence of a r.k. for F is expressed in terms of the evaluations of F . For any linear class F of functions on the basic set E to the range space R , where R may in general be any linear space, the evaluation of F at $y \in E$ is the linear transformation T_y of F into R defined by $T_y f = f(y)$.

If F is a normed linear space and R is a l.c.s then we shall say that F is a proper functional space (relative to R , whenever different range spaces are possible) if all of the evaluations are F^*-R^* -continuous.

If the evaluations are continuous in the weak topology of F and any topology in R which is stronger than $w(R, R^*)$ it follows that F is a proper functional space.

We remark that F is a proper functional space if and only if T_y^* exists for each $y \in E$. We are now in a position to state the existence theorem for the Hilbert space F .

Theorem 1. F has a r.k. if and only if it is a proper functional space.

Proof. If the r.k. exists then the defining properties 1) and 2) show that for any $y \in E$ the transformation which carries $r^* \in R^*$ into $K(\dot{x}, y)r^* \in F$ is the adjoint of T_y^* . Conversely, if T_y^* exists for each y then the family $T_x T_y^*$ of transformations of R^* into R has property 1) since the function $T_x T_y^* r^*$ is the element $T_y^* r^* \in F$, and property 2) holds because

$$\langle r^*, f(y) \rangle = \langle r^*, T_y f \rangle = (T_y^* r^*, f) = (T_x T_y^* r^*, f(\dot{x})) .$$

The structure of $K(x, y)$ as $T_x T_y^*$ shows that for each (x, y) it is a linear transformation. Moreover, the R^* -continuity of T_x and the R -continuity of T_y^* imply that the composition $K(x, y)$ is R - R^* continuous (taking for the common space F the weak topology, this is clear). If F is a proper functional space by virtue of the evaluations being continuous in the weak topology of F and a topology τ on R which is stronger than the $w(R, R^*)$ topology, then $K(x, y)$ is continuous in the topologies $w(R^*, R)$ and τ for each (x, y) .

In any case, $K(x, y)^*$ exists when K exists and (by means of the canonical identification) gives a transformation of R^* into R . For any $r^*, s^* \in R^*$ we have

$$\begin{aligned} \langle s^*, K(x, y)^* r^* \rangle &= \overline{\langle r^*, K(x, y) s^* \rangle} = \overline{\langle r^*, T_x T_y^* s^* \rangle} = \overline{(T_x^* r^*, T_y^* s^*)} \\ &= (T_y^* s^*, T_x^* r^*) = \langle s^*, K(y, x) r^* \rangle , \end{aligned}$$

i. e. $K(x, y)^* = K(y, x)$. In summary we have

Theorem 2. If F has a r.k. $K(x, y)$ then

(i) $K(x, y) = T_x T_y^*$,

(ii) $K(x, y)$ is linear and R - R^* continuous ,

(iii) $K(x, y)$ is $w(R^*, R) - \tau$ continuous whenever the evaluations are weak $-\tau$ continuous for any topology τ on R which is stronger than $w(R, R^*)$.

(iv) $K(x, y)^*$ exists and K has the symmetry property

$$K(x, y)^* = K(y, x) .$$

Some of the further basic properties of ordinary r.k.'s which carry over directly to the present case take the following forms:

1) The elements of F of the form $K(\dot{x}, y)r^* = T_y^* r^*$, $y \in E$ and $r^* \in R^*$ are complete in F .

2) If $\{\varphi_\alpha\}$ is a complete orthonormal system in F then for any $r^*, s^* \in R^*$,

$$\langle r^*, K(x, y)s^* \rangle = \sum_{\alpha} \langle r^*, \varphi_\alpha(x) \rangle \overline{\langle s^*, \varphi_\alpha(y) \rangle} ,$$

the series being absolutely convergent.

3) If F with r.k. K is a subspace of a larger Hilbert space \mathcal{H} then the formula

$$\langle r^*, f(y) \rangle = \langle K(\dot{x}, y)r^*, h \rangle$$

determines the projection f of the element $h \in \mathcal{H}$ on F .

4) If F_1 is a closed subspace of the space F with r.k. $K(x, y) = T_x T_y^*$ and P_1 is the projection on F_1 then the r.k. of F_1 is

$$K_1(x, y) = T_x P_1 T_y^* .$$

If F_2 is the orthogonal complement of F_1 in F then K is the sum of K_1 and K_2 .

5) $K(y, y)$ is positive for all $y \in E$ and, in fact,

$$\langle r^*, K(y, y)r^* \rangle = \|K(\dot{x}, y)r^*\|^2 .$$

In the case where R is a Hilbert space the r.k. $K(x, y)$ is a family

of bounded operators on R . If in addition it is known that the evaluations are completely continuous then the operators $K(x, y)$, as compositions of completely continuous transformations T_x with bounded transformations T_y^* , are also completely continuous.

For a complete orthonormal system $\{\rho_\alpha\}$ in R the Fourier expansion of $f(y)$ for any $f \in F$ and $y \in E$ takes the form

$$f(y) = \sum_{\alpha} (f, K(\dot{x}, y) \rho_{\alpha}) \rho_{\alpha}$$

by means of the reproducing property of K .

In case R is separable, and in particular for finite dimensional range spaces, we can take advantage of the fact that bounded operators on R have well determined matrix representations in terms of a complete orthonormal system $\{\rho_n\}$ ¹. In order to obtain such representations for the r.k. . In this way we obtain a matrix $\{k_{mn}(x, y)\}$ corresponding to K , whose elements

$$k_{mn}(x, y) = (K(x, y) \rho_m, \rho_n)$$

are complex valued functions of x and y satisfying the symmetry condition

$$k_{mn}(x, y) = \overline{k_{nm}(y, x)} .$$

It is easy to see that the diagonal elements are ordinary r.k.'s; in fact we shall see that $\langle r^*, K(x, y) r^* \rangle$ is such a kernel in the general case and we shall identify the space having this r.k. in a later section.

We conclude this section with a few examples of proper functional spaces of the type which we have introduced above.

Example 1. We consider the space F to be the space of all functions

1. See M. H. Stone [10] for a complete discussion of matrix representations of operators in Hilbert space.

defined on the rectangle $0 \leq x \leq a$, $0 \leq y \leq b$, and vanishing on the boundary, such that the norm given by the Dirichlet integral is finite, that is, for $h(x, y) \in F$

$$\|h(x, y)\|^2 = \int_0^b \int_0^a \left| \frac{\partial h}{\partial x} \right|^2 + \left| \frac{\partial h}{\partial y} \right|^2 dx dy .$$

This space can be completely characterized (see the forthcoming paper of N. Aronszajn and K. T. Smith, Theory of potentials) as follows: F is the space of all potentials of order 1 of L_2 functions, vanishing on the boundary.

We choose as basic set the interval $[0, a]$ of the x -axis and define the evaluation T_x by

$$T_x h = h(x, y) .$$

The class of all functions on $[0, b]$ of the form $h(x, y)$ can also be described as potentials (of order 1/2) of L_2 functions. These functions belong in particular to $L_2(0, b)$ and form a dense subspace there. Choosing R to be the space $L_2(0, b)$ our proper functional space is completely defined and has a r.k. $K(x, x')$ which is an operator in $L_2(0, b)$. If we let $G(x, y; x', y')$ denote the ordinary Green's function of the rectangle then K is the integral operator given by

$$K(x, x') f(y') = \int_0^b G(x, y; x', y') f(y) dy .$$

In this case the evaluations and the r.k. are completely continuous.

Example 2. Consider the space F of all harmonic functions in the unit circle under the norm given by the L_2 norm on the boundary. Let E be the open interval $(0, 1)$ and define $T_\rho u$ for $u \in F$ to be the function on the unit circumference which is equal to the restriction of u to the circle $|z| = \rho$, i.e. if $z = re^{i\theta}$ then $T_\rho u = u(\rho e^{i\theta})$. The range space R can be taken

to be the space of all continuous functions on $[0, 2\pi]$ under the upper bound norm. The functions $T_\rho u$ for $u \in F$ are dense in R . The conjugate space R^* is the space of all measures of finite total variation on $[0, 2\pi]$ and if $\mu \in R^*$, $f \in R$,

$$\langle \mu, f \rangle = \int_0^{2\pi} f d\mu.$$

The r.k. is described in terms of the Poisson kernel $P(z, \zeta)$ corresponding to a point z inside the unit circle and a point ζ on the boundary. Let $\zeta = e^{i\theta}$ and consider two interior points $z_1 = \rho_1 e^{i\varphi_1}$ and $z_2 = \rho_2 e^{i\varphi_2}$. Then $K(\rho_1, \rho_2)$, the r.k. of F_1 is given by

$$K(\rho_1, \rho_2) \mu(\varphi_2) = \int_0^{2\pi} \int_0^{2\pi} P(\rho_1 e^{i\varphi_1}, e^{i\theta}) P(\rho_2 e^{i\varphi_2}, e^{i\theta}) d\theta d\mu(\varphi_1).$$

§ 2. The tilde correspondence, positive matrices.

If F is a Hilbert space of functions on E to the l.c.s. R then we obtain from F a Hilbert space \tilde{F} of complex valued functions on $E \times R^*$ as follows. \tilde{F} is the class of all functions of the form

$$\tilde{f}(x, r^*) = \overline{\langle r^*, f(x) \rangle} \quad \text{for } f \in F,$$

with the norm

$$\|\tilde{f}\| = \|f\|.$$

The correspondence between the functions of f and \tilde{f} is clearly linear and it is one-one since R^* distinguishes elements of R . Hence the norm can be transferred in this manner and \tilde{F} is a well defined Hilbert space isomorphic to F . We remark that for each x , $\tilde{f}(x, r^*)$ is a weak star continuous anti-linear functional on R^* ; in fact, it is the element $f(x) \in R$.

On the other hand, suppose we are given a Hilbert space \tilde{F} of complex valued functions on a set \tilde{E} and suppose that \tilde{E} can be interpreted as $E \times R^*$ for some set E and l.c.s. R , and suppose further that for each $\tilde{f} \in \tilde{F}$ and $x \in E$, $\tilde{f}(x, r^*)$ is a weak star continuous anti-linear functional on R^* . Then a Hilbert space F of functions on E to R is obtained from \tilde{F} by reversing the above procedure. Namely, to $\tilde{f} \in \tilde{F}$ associate the function on E to R , $f(x)$, whose value at x is the element $\tilde{f}(x, r^*) \in R$ and put $\|f\| = \|\tilde{f}\|$.

The correspondence thus established between spaces F and spaces \tilde{F} is called the tilde correspondence.

Theorem 1. F is a proper functional space if and only if \tilde{F} is a proper functional space.

Proof. The latter condition means that for all $(x, r^*) \in E \times R^*$,

$\tilde{f}_n \rightarrow \tilde{f}$ strongly in \tilde{F} implies $\tilde{f}_n(x, r^*) \rightarrow \tilde{f}(x, r^*)$. For the corresponding elements of F this means that $f_n \rightarrow f$ strongly in F implies $\langle r^*, f_n(x) \rangle \rightarrow \langle r^*, f(x) \rangle$, for all $x \in E$, $r^* \in R^*$. But this is exactly the R^* -continuity of the evaluations of F as desired.

If F has r.k. $K(x, y)$ then the r.k. of \tilde{F} is given by

$$\tilde{K}(x, r^*; y, s^*) = \overline{\langle r^*, K(x, y)s^* \rangle}.$$

It is clear that for fixed (y, s^*) , $\tilde{K} \in \tilde{F}$ as a function of (x, r^*) ; in fact, it corresponds to the function $K(\dot{x}, y)s^* \in F$. Moreover,

$$\begin{aligned} \tilde{f}(y, s^*) &= \overline{\langle s^*, f(y) \rangle} = \overline{\langle K(\dot{x}, y)s^*, f(\dot{x}) \rangle} \\ &= \langle f(\dot{x}), K(\dot{x}, y)s^* \rangle \\ &= \langle \tilde{f}(\dot{x}, \dot{r}^*), \widetilde{K(\dot{x}, y)s^*} \rangle, \text{ by isometry,} \\ &= \langle \tilde{f}(\dot{x}, \dot{r}^*), \overline{\langle \dot{r}^*, K(\dot{x}, y)s^* \rangle} \rangle \\ &= \tilde{f}(\dot{x}, \dot{r}^*), \tilde{K}(\dot{x}, \dot{r}^*; y, s^*) \rangle. \end{aligned}$$

By means of the tilde correspondence it is possible to interpret some of the theorems of the ordinary theory directly as theorems in the present setting and when this is possible we shall limit our discussion of the result accordingly. However, in most cases it is necessary to enter into additional considerations either in order to supply the proper interpretation of the space \tilde{F} which results from the usual theorem or in order to investigate those aspects of the theorem which have special significance for the spaces and kernels which we are studying.

Definition. A family $K(x, y)$ of linear R - R^* continuous transformations of R^* into R defined for each $x, y \in E$, where R is any l.c.s., is a positive matrix relative to R (p.m. rel. R) if for each choice of $y_1, y_2, \dots, y_n \in E$ and

$$r_1^*, r_2^*, \dots, r_n^* \in R^*,$$

$$\sum_{i=1}^n \sum_{j=1}^n \langle r_i^*, K(y_i, y_j) r_j^* \rangle \geq 0.$$

The tilde correspondence or a direct computation gives

Theorem 2. If $K(x, y)$ is the r.k. of the space F of functions on E to R then $K(x, y)$ is a p.m. rel. R .

In contrast to the usual theory the converse of this theorem is not always true. The precise statement of the circumstances in which a given p.m. determines a p.f.s. for which it is the r.k. is given in Section 5 in connection with the general theory of functional completion of spaces of the type considered here. We need only remark at this point that if such a p.f.s. exists for a p.m. rel. R then it is uniquely determined by the p.m. and the space R . The proof of this fact is obtained by means of the tilde correspondence.

Remark 1. We may change R in certain ways so as to obtain a p.f.s. and r.k. which are identified in a canonical way with the original ones but according to our definitions they must be considered as distinct.

Remark 2. The definition of p.m. in the usual theory involves complex coefficients $\xi_1, \xi_2, \dots, \xi_n$ which are absorbed in the functionals r_j^* in the present case.

§ 3. Isomorphism, equivalence and subordination.

In this section we consider functional Hilbert spaces F and G with the same basic set E but with different range spaces, R and S respectively. We shall also consider p.m.'s rel. R and S . As a matter of notational convenience we shall use $K(x, y)$ to denote a p.m. rel. R and $L(x, y)$ to denote a p.m. rel. S and if F and G are proper functional spaces their r.k.'s will be K and L respectively.

The spaces F and G are essentially the same, as functional spaces, whenever R and S as well as F and G are isomorphic in the sense of their respective structures and in such a way that the operation of "evaluation at x " is preserved by the isomorphisms. The natural notion of essential identity of K and L would require that R and S be isomorphic, in which case R^* and S^* are isomorphic under the adjoint transformation and the weak star topologies, and that for each (x, y) , $K(x, y)$, and $L(x, y)$ transform corresponding elements into corresponding elements. Before proceeding with the precise formulations of these ideas and the examination of further questions to which they give rise, we make a definition for general linear spaces F and G of functions on E to linear spaces R and S respectively. Namely, if σ is a linear transformation of R into S , the transformation T_σ induced by σ is the linear transformation of F into G defined by

$$T_\sigma f = g \text{ whenever } g(x) = \sigma f(x) \text{ for all } x.$$

With F and G Hilbert spaces we make the

Definition 1. F is isomorphic to G , $F \cong G$, if there exists a linear homeomorphism σ of R onto S such that T_σ is an isometric isomorphism

of F onto G.

Definition 2. $K(x, y)$ is equivalent to $L(x, y)$, $K \cong L$, if there is a linear homeomorphism σ of R onto S such that for all (x, y)

$$L(x, y) = \sigma K(x, y) \sigma^* .$$

Theorem 1. If F is a p.f.s.rel. R and $G \cong F$ then G is a p.f.s. rel. S and $K \cong L$ under the same linear homeomorphism σ .

Proof. We must show that $\sigma K(x, y) \sigma^*$ is a r.k. for G. Since for any $s^* \in S^*$ and any $y \in E$,

$$\sigma K(x, y) \sigma^* s^* = \sigma (K(x, y) \sigma^* s^*) \quad \text{for all } x ,$$

it follows that

$$\sigma K(\dot{x}, y) \sigma^* s^* = T_{\sigma}(K(\dot{x}, y) \sigma^* s^*) .$$

The result follows since any r^* has the form $\sigma^* s^*$ and T_{σ} is an isometry.

Theorem 2. If $K \cong L$ then $F \cong G$.

Proof. If T_{σ} is the transformation of F into the space of all functions on E to S induced by σ then T_{σ} is one-one and its range is a Hilbert space isomorphic to F under the norm $\|T_{\sigma}f\| = \|f\|$. This space has r.k. L by Theorem 1 and hence must be the space G.

Theorem 3. If $K(x, y)$ is a p.m.rel. R and σ is a continuous linear transformation of R into a l.c.s. S then $L(x, y) = \sigma K(x, y) \sigma^*$ is a p.m. rel. S. If σ is a homeomorphism of R onto S then $L \cong K$.

The theorem is immediate.

We shall examine now the r.k.'s generated by a given kernel and a continuous linear mapping in this way.

Definition 3. $L(x, y)$ is subordinate to $K(x, y)$, $L \text{ sub } K$, if

$L = \sigma K \sigma^*$ for some continuous linear transformation σ of R into S .

If $L \text{ sub } K$ and T_σ denotes the transformation of F into the space of all functions on E to S induced by σ , we have

Theorem 4. If $L \text{ sub } K$ then G is the range of T_σ with the norm

$$\|g\|^2 = \inf_{T_\sigma f = g} \|f\|^2.$$

Alternatively, G is the isometric image by T_σ of the orthogonal complement in F of the nullspace of T_σ , with the evaluations σT_x .

Proof. Let \mathfrak{n} denote the nullspace of T_σ . Its orthogonal complement \mathfrak{n}^\perp is a closed subspace of F and for $f_1 \in \mathfrak{n}^\perp$

$$\|f_1\| = \inf_{f - f_1 \in \mathfrak{n}} \|f\|.$$

The restriction of T_σ to \mathfrak{n}^\perp is an isomorphism of \mathfrak{n}^\perp onto the range M of T_σ and it is clearly also an isometry if M has the norm specified in the theorem.

To show that the Hilbert space M has the r.k. $L(x, y)$, i. e. $G = M$, consider first $L(\dot{x}, y)s^*$ for any $s^* \in S^*$ and any $y \in E$:

$$L(x, y)s^* = \sigma K(x, y)\sigma^*s^* \quad \text{for all } x$$

and $K(\dot{x}, y)\sigma^*s^* \in F$. Hence

$$L(\dot{x}, y)s^* = T_\sigma K(\dot{x}, y)\sigma^*s^* \in M.$$

For the reproducing property we must compute $(L(\dot{x}, y)s^*, g)$ for any $g \in M$. This is done by invoking the isometry of M with \mathfrak{n}^\perp and computing (f_1, f_2) where f_1 and f_2 are the elements of \mathfrak{n}^\perp corresponding to $L(\dot{x}, y)s^*$ and g respectively. We assert that $K(\dot{x}, y)\sigma^*s^* \in \mathfrak{n}^\perp$ from which it follows that this element is f_1 and finally that

$(L(\dot{x}, y)s^*, g) = (K(\dot{x}, y)\sigma^*s^*, f_2) = \langle \sigma^*s^*, f_2(y) \rangle = \langle s^*, \sigma f_2(y) \rangle = \langle s^*, g(y) \rangle$,
the reproducing property.

To complete the proof we note that $f \in \mathcal{N}^\perp$ if and only if $(f, f_0) = 0$ whenever $\sigma f_0(x) = 0$ for all x . Applying this to $K(\dot{x}, y)\sigma^*s^*$ we find

$$(K(\dot{x}, y)\sigma^*s^*, f_0) = \langle \sigma^*s^*, f_0(y) \rangle = \langle s^*, \sigma f_0(y) \rangle = 0$$

and our assertion follows.

This theorem motivates the definition:

Definition 4. G is subordinate to F , $G \text{ sub } F$, if there is a continuous linear transformation σ of R into S such that G is isometrically isomorphic by T_σ to the complement of the nullspace of T_σ .

If $G \text{ sub } F$ then the restriction of T_σ is clearly such an isometric isomorphism and conversely. Therefore, Theorem 4 has as a corollary

Theorem 5. $L \text{ sub } K$ implies $G \text{ sub } F$.

Conversely,

Theorem 6. If $G \text{ sub } F$ and F is a p.f.s. then G is a p.f.s. and
 $L \text{ sub } K$.

Proof. Let U_x denote the evaluation of G at x and denote by U the inverse of the restriction of T_σ to \mathcal{N}^\perp . Then

$$U_x g = g(x) = \sigma(Ug)(x) = \sigma T_x U g \quad \text{for all } g, \quad \text{i.e.}$$

$$U_x = \sigma T_x U.$$

It follows that the U_x 's are S^* -continuous, giving the first assertion. Computation of $U_x U_y^*$ completes the proof.

Applying the foregoing theorems to special choices of σ leads to the

answers to some natural questions regarding the dependence of the space F and its r.k. K on the range space R .

1) If R is a closed subspace of a l.c.s. S then K has a natural extension to a p.m.rel. the larger space, given by $\sigma K\sigma^*$ where σ is the injection map of R into S . The space which has r.k. $\sigma K\sigma^*$ is F considered as a space of functions on E to S .

2) If R_0 is a closed subspace of R and σ is the quotient map of R onto $S = R/R_0$ then the p.m.rel. S , $\sigma K\sigma^*$, the quotient kernel of $K \bmod R_0$ is the r.k. of the space $F \ominus F(E, R_0)$ under the evaluations σT_x , where $F(E, R_0)$ is the subspace of F of all functions which map E into R_0 .

3) If R is a l.c.s. under a second topology weaker than its given topology, S denotes R under the weaker topology and σ is the identity as a mapping of R onto S then σ^* is the injection of S^* into R^* and $\sigma K\sigma^*$ is the r.k. of F as a p.f.s.rel. S . We may call this r.k. the topological restriction of K .

If F and G are isomorphic proper functional spaces there may be several linear homeomorphisms of R onto S with the properties required by the definition. If σ_1 and σ_2 are two such then $\lambda = \sigma_1^{-1}\sigma_2$ is an invariant automorphism in R , that is, a linear homeomorphism of R onto R which induces an isometric isomorphism in F . It is clear that the invariant automorphisms form a group. In the case of ordinary r.k.'s the automorphism of the complex plane consisting of multiplication by $e^{i\theta}$, $0 \leq \theta \leq 2\pi$ is clearly an invariant automorphism and because of the isometry requirement there can be no others. Trivial automorphisms of this type, i.e.

automorphisms given by multiples $e^{i\theta} I$ of the identity I , will always be invariant. An example of a case where non-trivial invariant automorphisms exist is the following. Let F_1, F_2, \dots, F_n be spaces of complex valued functions having ordinary r.k.'s $K_1(x, y), K_2(x, y), \dots, K_n(x, y)$. Consider as F the direct sum

$$F = F_1 \dot{+} F_2 \dot{+} \dots \dot{+} F_n .$$

The elements of F are n -tuples $f = (f_1, f_2, \dots, f_n)$, $f_i \in F_i$, and the norm is given by

$$\|f\|^2 = \sum_{i=1}^n \|f_i\|_i^2 .$$

F is a proper functional space with unitary n -space as the range space.

If $f \in F$ then $f(y) = (f_1(y), f_2(y), \dots, f_n(y))$ and if $r = (r_1, r_2, \dots, r_n)$

$$(r, f(y)) = \sum_{i=1}^n r_i \overline{f_i(y)} .$$

Letting $K(x, y)$ stand for the matrix $\{K_i(x, y) \delta_{ij}\}$,

$$K(\dot{x}, y)r = (K_1(\dot{x}, y)r_1, K_2(\dot{x}, y)r_2, \dots, K_n(\dot{x}, y)r_n) \in F$$

and
$$(K(\dot{x}, y)r, f) = \sum_{i=1}^n (K_i(\dot{x}, y)r_i, f_i)_i = \sum_{i=1}^n r_i \overline{f_i(y)} .$$

Hence $K(x, y)$ is the r.k. of F .

All automorphisms of the form $\sigma = \{e^{i\theta_k} \delta_{k\ell}\}$, θ_k real, are invariant as is seen by direct computation of either $\|T_\sigma f\|$, $T_\sigma f = (e^{i\theta_1} f_1, \dots, e^{i\theta_n} f_n)$, or $\sigma K(x, y) \sigma^*$.

If $A = \{a_{k\ell}\}$ is any automorphism of n -space and we compute $AK(x, y)A^*$ we arrive at the matrix $\left\{ \sum_j a_{kj} \overline{a_{\ell j}} K_j \right\}$. In the case where all spaces F_i are the same, this gives a necessary and sufficient condition for

A to be invariant, namely $\sum_j a_{kj} \bar{a}_{lj} = \delta_{kl}$, i.e. that A be unitary.

An example where the range space is n-dimensional and the only invariant automorphisms are the trivial ones is obtained by taking F to be the space of all functions on the interval $[0,1]$ which have (n-1) derivatives continuous and n-th derivative absolutely continuous and belonging to $L_2(0,1)$. The norm in F is taken to be

$$\|f\|^2 = \int_0^1 [|f|^2 + |f'|^2 + \dots |f^{(n-1)}|^2] dx .$$

F is exactly the class of functions for which this norm is finite. Defining the evaluation T_x to be the transformation of F into n-space given by

$$T_x f = (f(x), f'(x), \dots, f^{(n-1)}(x))$$

we obtain a proper functional space having the required properties.

§4. Restrictions. Self adjoint and positive r.k.'s.

Applying the tilde correspondence and the usual theorem on restrictions of r.k.'s to subsets of their basic sets we find two cases of interest. Namely, when \tilde{K} is restricted to a subset $E_1 \subset E$ and R^* remains fixed, and when $E_1 = E$ but R^* is restricted to a subspace R_1^* of R^* . The first case gives the direct analogue of the usual theorem.

Theorem 1. If $E_1 \subset E$ and K is a p.m.rel. R defined in E then the restriction K_1 of K to E_1 is a p.m.rel. R and is the r.k. of the space F_1 of all restrictions to E_1 of functions in the space F having r.k. K , and the norm in F_1 is given by

$$\|f_1\| = \inf \|f\| ,$$

the infimum being taken over all $f \in F$ whose restriction to E_1 is f_1 .

In the second case, let σ^* denote the injection map of R_1^* into R^* . Since σ^* is continuous in the weak star topology, its adjoint exists and maps R onto the conjugate S of R_1^* . Calling this transformation σ and taking S in the $w(S, R_1^*)$ topology so that $S^* = R_1^*$, it becomes clear that the meaning to be assigned to the restriction of K to R_1^* is the p.m. rel. S given by $\sigma K \sigma^*$. In order to identify the space S in terms of R and determine the functional space having the restriction as r.k. we analyze separately the case 1, where R_1^* is weak star dense in R^* , and the case 2, where R_1^* is a closed subspace of R^* . The general result follows by performing the restriction first to R_1^* from the weak star closure of R_1^* , applying case 1, and then, applying case 2, from R^* to the closure.

The transformation σ assigns to $r \in R$ its restriction to R_1^* and hence in case 1 it is one-one, R and S are equal, and σ gives the

identity mapping of R under the $w(R, R^*)$ topology (or under its given topology) onto R under the weaker topology $w(R, R_1^*)$. It is now apparent that $\sigma K \sigma^*$ is the topological restriction of K induced by this change of topologies and therefore that the corresponding space is F considered as a space of functions into R with the weaker topology.

In case 2, we make use of the nullspace \mathcal{N}_σ of σ , that is, the polar of R_1^* as a subspace of R . Knowing that R_1^* is closed we can conclude from the general result cited in the preliminaries that S is isomorphic to the quotient space R/\mathcal{N}_σ and in fact homeomorphic when S has the $w(S, R_1^*)$ topology and R/\mathcal{N}_σ has the quotient topology induced by $w(R, R^*)$. If τ denotes this isomorphism and q the quotient map then one verifies immediately that $\sigma = \tau q$. Consequently, $\sigma K \sigma^* = \tau(q K q^*)\tau^*$ and the restriction is seen to be isomorphic to the quotient kernel of $K \bmod \mathcal{N}_\sigma = R_1^{*\perp}$. The space having the restriction as r.k. is therefore $F \ominus F(E, R_1^{*\perp})$ under the evaluations σT_x .

We are now in a position to state

Theorem 2. The restriction of K to a subspace R_1^* of R^* is isomorphic to the topological restriction of the quotient kernel of $K \bmod R_1^{*\perp}$ induced by the weakening of the topology in $R/R_1^{*\perp}$ due to the weakening of the topology in R from the given topology to $w(R, R_1^*)$. The restriction is the r.k. of the space $F \ominus F(E, R_1^{*\perp})$ under the evaluations given by T_x followed by the appropriate canonical mappings.

The definitions of self adjoint and positive transformations of R^* into R given in the preliminaries show that these properties for a r.k. K are determined by the behavior of the ordinary r.k.'s $\langle r^*, K(x, y)r^* \rangle$.

An ordinary r.k. $K(x, y)$ can be thought of as a mapping

$$y \longrightarrow K_y = K(\dot{x}, y)$$

of a set E onto a complete set of elements in a Hilbert space. On the other hand, whenever a mapping of a set E into an abstract Hilbert space is encountered we may derive from it an ordinary r.k. . Suppose $y \longrightarrow K_y$ is such a mapping of E into H . Then the closed subspace F_0 of H which is generated by the elements K_y for $y \in E$ becomes a proper functional space under the evaluations $T_x f = (f, K_x)$ and the r.k. is $K(x, y) = (K_y, K_x)$.

In our present considerations we have for any $r^* \in R^*$ the mapping

$$y \longrightarrow T_y^* r^*$$

of E into F . Letting F_{r^*} be the closed subspace of F generated by these elements, the resulting proper functional space consists of all complex valued functions of the form

$$f'(x) = (f, T_x^* r^*) = \overline{\langle r^*, f(x) \rangle}$$

and the r.k. is

$$(T_y^* r^*, T_x^* r^*) = \overline{\langle r^*, K(x, y) r^* \rangle}.$$

We wish to distinguish between the subspace F_{r^*} of F and the space of functions so obtained from it. The notation \tilde{F}_{r^*} for the latter is suggested since there is a connection with the tilde correspondence, as we shall see.

It is immediately seen that F_{r^*} can be written as $F \ominus F(E, \mathfrak{n}_{r^*})$ where \mathfrak{n}_{r^*} is the nullspace in R of the functional r^* . If $[r^*]$ denotes the one-dimensional subspace of R^* generated by r^* , then $\mathfrak{n}_{r^*} = [r^*]^\perp$, R/\mathfrak{n}_{r^*} is a one-dimensional space and the restriction of $K(x, y)$ to $[r^*]$ is isomorphic to the quotient kernel of $K \bmod \mathfrak{n}_{r^*}$. The latter is a p.m. rel. the one-dimensional space R/\mathfrak{n}_{r^*} and is the r.k. of $F \ominus F(E, \mathfrak{n}_{r^*})$ under the evaluations $q T_x$ where q is the quotient map. If we interpret

the elements of R/\mathfrak{n}_{r^*} as complex numbers by assigning to each element r the number $\langle \overline{r^*}, r \rangle$ for the fixed r^* with which we started, we again obtain the space \tilde{F}_{r^*} .

The connection with the tilde correspondence is now clear. The above quotient kernel gives, under the tilde correspondence, an ordinary proper functional space on $E \times [r^*]$ and \tilde{F}_{r^*} is obtained from this space by fixing the element of $[r^*]$ as r^* .

Notice that when R is a separable Hilbert space and the r.k. is expanded into a matrix following a complete orthonormal sequence $\{\rho_n\}$, the spaces whose r.k.'s are the diagonal elements are just the spaces \tilde{F}_{ρ_n} .

Returning to the question of self adjointness and positiveness of $K(x, y)$ it is clear now that the condition on the space F in order that K have one of these properties is that each of its subspaces F_{r^*} have a corresponding property as determined by the conditions for reality and positiveness of ordinary r.k.'s. We summarize with the statements:

Theorem 3. $K(x, y)$ is self adjoint for all $x, y \in E$ if and only if for every $r^* \in R^*$, $f \in F_{r^*}$ implies the existence of an $\tilde{f} \in F_{r^*}$ such that
 $\langle r^*, f(x) \rangle = \langle r^*, \tilde{f}(x) \rangle$ for all x , and $\|\tilde{f}\| = \|f\|$.

Remark. Since $\langle r^*, f(x) \rangle$ is a well determined complex valued function on E there can be at most one function $\tilde{f}(x)$ in F_{r^*} giving it a representation as $\langle r^*, \tilde{f}(x) \rangle$.

Theorem 4. ^{1.} $K(x, y)$ is nonnegative for all $x, y \in E$ if and only if it is self adjoint for all x, y and for every $r^* \in R^*$, $f = \tilde{f}$ in F_{r^*} implies the existence of an $\tilde{f} \in F_{r^*}$ such that $\|\tilde{f}\| \leq \|f\|$ and $\langle r^*, \tilde{f}(x) \rangle \geq |\langle r^*, f(x) \rangle|$

1. For the corresponding theorem for the usual r.k.'s see N. Aronszajn and K. T. Smith [4].

for all x .

We remark next that in view of the symmetry property of a r.k. $K(x, y)$ the self adjointness is equivalent to the condition $K(x, y) = K(y, x)$.

We conclude this section by examining a property of $K(x, y)$ which corresponds to the property of an ordinary proper functional space, that $f \in F$ imply $\bar{f} \in F$ and $\|f\| = \|\bar{f}\|$, that is, the condition that the r.k. of F be real.

Define a conjugation J in R to be a continuous mapping of R onto itself which is antilinear and involutory, i.e., $J^2 = \text{the identity in } R$. The conjugation J in R induces a conjugation J_* in R^* by the formula

$$\langle r^*, Jr \rangle = \overline{\langle J_* r^*, r \rangle}.$$

The transformation J induces a transformation T_J of F onto a space F_J of functions on E to R in the usual way, namely,

$$(T_J f)(x) = Jf(x), \quad \text{for all } x.$$

The space F_J can be normed by putting $\|T_J f\| = \|f\|$. With this norm F_J is a p.f.s. rel. R and the r.k. is easily seen to be $JK(x, y)J_*$.

If we define $K(x, y)$ to be invariant under the conjugation J whenever

$$K(x, y) = JK(x, y)J_*$$

then it is clear that a necessary and sufficient condition for K to be invariant is the coincidence of the spaces F and \bar{F} . We state this in the form suggestive of the condition for reality of an ordinary r.k., namely,

Theorem 5. K is invariant under a conjugation J if and only if $f \in F$ implies $T_J f \in F$ and $\|T_J f\| = \|f\|$.

§5. Functional completion, admissible spaces.

In this section we shall consider several related questions associated with the general problem of constructing proper functional spaces.

We begin by considering a normed linear space F of functions on a set E with values in a linear space V . F will be said to be admissible if there is a topology for V making it a l.c.s. and such that F with this range space is a p.f.s., i.e. the evaluations are continuous in this topology of V . Any such topology will be called an admissible topology for V .

Since there is a strongest topology for V which makes the evaluations continuous, the conjugate V_M^* of V with this topology is the maximal subspace of V' which can serve to define topologies on V which are admissible. It is clear, therefore, that F is admissible if and only if V_M^* distinguishes points of V , or equivalently, weak-star dense in V' . The space V_M^* consists of all functionals on V whose composition with each of the evaluations gives a continuous functional on F . The collection of all admissible topologies for V in case of an admissible space is described as consisting of all topologies between the linear and the Mackey determined by subspaces of V_M^* which are weak-star dense in V' .

It is often convenient to replace a given functional space with a range space V by a second functional space consisting of the same class F and with the range space V_0 consisting of the exact range of F , that is, the subspace of V generated by the elements which are actually assumed as values by some function in F .

The second question which we take up here is one which arises in applying the tilde correspondence. When some of the theorems of the ordinary

theory are applied to the corresponding problems in the present case by means of the tilde correspondence the configuration which is directly obtained is an incomplete space \tilde{F} of functions on a set $E \times R^*$ for some linear space R^* and the problem arises of determining the space R appropriately.

Let us consider in general a normed linear proper functional space \tilde{F} of functions on $E \times R^*$ which are antilinear in their dependence on r^* . The tilde correspondence carries over to general normed linear spaces so we may consider this more general setting. The space R must be chosen so that its conjugate is R^* and hence it will be a subspace of $(R^*)'$ with the $w(R, R^*)$ topology. Now, the choice of subspaces of $(R^*)'$ corresponds to the choice of topologies in R^* and since R must contain the elements $\tilde{f}(y, r^*)$ for each $y \in E$ and $\tilde{f} \in \tilde{F}$ we see that there is a minimal R , say R_0 , and a minimal topology on R^* , $w(R^*, R_0)$. However, this topology may fail to be a Hausdorff topology since R_0 may not distinguish elements of R^* . If \mathcal{N} denotes the zero subspace, R_0^\perp , of this topology then we consider the quotient space R^*/\mathcal{N} . R_0 is the exact range of the space F corresponding to \tilde{F} and any stronger topology on R^* having the same zero subspace \mathcal{N} determines a completion of R_0 .

We introduce a particular stronger topology determined by the space \tilde{F} as follows. For each fixed choice of y and r^* , $\tilde{f}(y, r^*)$ is a linear functional on \tilde{F} , namely, the evaluation of \tilde{F} , and hence $\tilde{f}(y, r^*) \in \tilde{F}^*$. Fixing y we thus obtain a linear transformation

$$r^* \longrightarrow \overline{\tilde{f}(y, r^*)}$$

of R^* into \tilde{F}^* . The definition

$$\|r^*\|_y = \overline{\|\tilde{f}(y, r^*)\|}^*$$

determines a pseudo norm on R^* for each y . The desired topology on R^* is the locally convex topology defined there by this family of pseudo-norms, i. e. the neighborhoods of zero are given by the subsets of all r^* such that $\|r^*\|_y < \alpha$ for some y and some $\alpha > 0$, and by finite intersections of such sets.

This topology will be denoted $\tilde{\tau}$.

Theorem 1. $\tilde{\tau}$ is stronger than $w(R^*, R_0)$, and has the same zero subspace \mathcal{N} .

Proof. We must show that each of the functionals $\tilde{f}(y, r^*)$ is continuous in $\tilde{\tau}$. Since

$$\|r^*\|_y = \sup_{\tilde{g} \in \tilde{F}} \frac{|\tilde{g}(y, r^*)|}{\|\tilde{g}\|} \geq \frac{|\tilde{f}(y, r^*)|}{\|\tilde{f}\|}$$

we have, for any $\epsilon > 0$,

$$|\tilde{f}(y, r^*)| < \epsilon \quad \text{whenever} \quad \|r^*\|_y < \frac{\epsilon}{\|\tilde{f}\|}.$$

This shows also that the zero subspace rel. $\tilde{\tau}$ is contained in the one rel $w(R^*, R_0)$; that the former contains the latter is obvious from the definition of $\|r^*\|_y$.

Theorem 2. If \tilde{F} is a reflexive Banach space then $\tilde{\tau}$ is weaker than the Mackey topology $m(R^*, R_0)$. Hence R_0 is the conjugate of R^* with the $\tilde{\tau}$ topology in this case.

Proof. It is enough to show that any neighborhood of zero of the form $\|r^*\|_y < \alpha$ contains a neighborhood of zero in the Mackey topology.

Consider the set $N_\alpha \subset R^*$ of all r_1^* such that $|\langle r_1^*, r \rangle| = |\tilde{f}(y, r_1^*)| < \alpha$ for all $r = \tilde{f}(y, \hat{r}^*)$ such that $\|\tilde{f}\| \leq 1$. The unit sphere $\|\tilde{f}\| \leq 1$ is weakly compact by our hypothesis and since the mapping of \tilde{F} into R_0 which maps \tilde{f} onto $\tilde{f}(y, \hat{r}^*)$ is continuous, it follows that the image of the unit sphere is compact and hence that N_α is a Mackey neighborhood of zero. Finally, for $r_1^* \in N_\alpha$

$$\sup_{\|\tilde{f}\| \leq 1} |\tilde{f}(y, r_1^*)| < \alpha,$$

and consequently $\|r_1^*\|_y < \alpha$.

As a special case of this theorem we obtain a fact which will be useful in applying the tilde correspondence to our general theory. This is the case in which \tilde{F} is obtained as the proper functional completion in the ordinary sense, of an incomplete Hilbert space \tilde{F}_0 on the basic set $E \times R^*$. The $\tilde{\tau}$ topology defined for the completion \tilde{F} gives us a well determined range space R_0 with respect to which the tilde correspondence can be applied to \tilde{F} and a Hilbert space of functions on E to R_0 obtained. In addition to the fact that R_0 is arrived at by a constructive procedure, it has the advantage of being the minimal range space for the completion.

We consider the general question of the possibility of functional completion of an incomplete Hilbert space F , of functions on E to a l.c.s. R , which is a p.f.s. rel. R .

Definition 1. A (proper) functional completion of F is a proper functional Hilbert space \bar{F} relative to a l.c.s. \bar{R} such that F with its norm is a dense subspace of \bar{F} and R with its given topology is a dense subspace of \bar{R} .

Theorem 3. A necessary and sufficient condition for a functional completion of F to exist is that for every Cauchy sequence $\{f_n\}$ in F

$$1) \quad f_n(x) \rightarrow 0 \text{ in } w(R, R^*), \text{ for all } x, \text{ implies } \|f_n\| \rightarrow 0.$$

When the condition is satisfied the completion is possible with respect to a minimal range space \bar{R} in the sense that any other subspace of $(R^*)'$ having a topology which induces the given topology in R and for which the completion exists must contain \bar{R} as a subspace.

Proof. The condition 1) is exactly the usual condition in order that \tilde{F} have a functional completion $\bar{\tilde{F}}$. This proves the necessity. To construct the completion we consider the class of all functionals on R^* of the form $f(x, r^*)$ for $f \in \tilde{F}$ and $x \in E$. Let R_0 denote the subspace which they generate in $(R^*)'$. If $(R^*)'$ is considered with its weak-star topology and R is considered with $w(R, R^*)$ then R is a subspace of $(R^*)'$.

The space R_0 can be obtained more directly prior to the completion of \tilde{F} by forming the topology $\tilde{\tau}$ on R^* constructed by means of \tilde{F} and taking the conjugate of R^* in this topology.

The smallest subspace of $(R^*)'$ containing both R and R_0 is $\bar{R} = R + R_0$. By a general remark in the preliminary section we may topologize \bar{R} in such a way that the induced topology in R is its given topology and $\bar{R}^* = R^*$. R is then dense in \bar{R} . Finally, the elements of $\bar{\tilde{F}}$ are all weak-star continuous antilinear functionals on $\bar{R}^* = R^*$ and the tilde correspondence can be applied to obtain the required functional completion of F .

Theorem 4. If R is sequentially complete in its topology and the condition 1) of Theorem 3 is satisfied then a unique functional completion

of F exists such that the range space is R .

Proof. Under the hypotheses of the theorem it is possible to extend the evaluations of F continuously to the abstract completion of F , making it into a proper functional space.

In particular, if R is a Banach space the theorem applies.

We are now in a position to state the form of the converse of Theorem 2, Section 2, which is true in the present theory.

Theorem 5. If $K(x, y)$ is a p.m. rel. R then there is a completion \bar{R} of R (and a well determined minimal one) corresponding to a topology on R^* stronger than $w(R^*, R)$ such that, considering K as a p.m. rel. \bar{R} , there exists a p.f.s. F rel. \bar{R} having K as r.k. .

If R is sequentially complete there exists a p.f.s. F rel. R having r.k. K .

This is proved directly by the tilde correspondence, using the above theorems, in view of the fact that, as in the usual theory, the construction of F depends on the functional completion of the class of functions of the form $\sum K(\dot{x}, y_i) r_i^*$ with the norm given by

$$\left\| \sum K(\dot{x}, y_i) r_i^* \right\|^2 = \sum \sum \langle r_j^*, K(y_j, y_i) r_i^* \rangle .$$

The following example shows that the exact converse of Theorem 2, Section 2, is false.

Let E consist of a single point x , H be a Hilbert space and R a dense subspace of H . We define a linear topology on R , by choosing $R^*=R$ (i.e. R^* is composed of functionals (x', \dot{r}) , $r' \in R$). As $K(x, y)$ we

choose the identity mapping of $R^* = R$ onto R . It is obvious that it is a p.m. rel. R . It is seen immediately that a corresponding p.f.s. F would have to have the whole of H for its range space (the evaluation T_x being an isometry onto H).

We conclude with a few more examples concerning the contents of this section.

First, an example of a space which is not admissible is provided by any Hilbert space of complex valued functions which is not a proper functional space. Such a space can be constructed from $L_2(0,1)$ by making use of the axiom of choice to select from the classes of equivalence (of functions defined and equal almost everywhere) functions which are defined everywhere and which form a linear space of functions.

Next, let F be the space of all functions defined on the rectangle $0 \leq x \leq a$, $0 \leq y \leq b$, vanishing on the boundary and having continuous first partial derivatives, with the Dirichlet integral as norm. The evaluations are defined as before by

$$T_x h(x, y) = h(x, \dot{y}) ,$$

in which case the exact range of F is the space R_0 of all functions on $[0, b]$ vanishing at the end points and having continuous first derivatives.

The topology for R_0 given by the norm

$$\|f\| = \sup_{0 \leq y \leq b} |f'(y)|$$

would not be an admissible topology for F . The topology given by the $L_2(0, b)$ norm is admissible and if the $\tilde{\tau}$ topology for $R^* = L_2(0, b)$ were constructed making use of the r.k. for the completion of F given in §1 it would be found that the completion \overline{R} of R is the space of all potentials of order $1/2$ of L_2 functions on $[0; b]$.

§6. Sums and differences.

If $K_1(x, y)$ and $K_2(x, y)$ are p.m.'s rel. R_1 and R_2 respectively and F_1 and F_2 are the spaces having these r.k.'s then by means of the tilde correspondence we shall study the possibilities of combining K_1 and K_2 to obtain other p.m.'s and determine their corresponding spaces.

Sums. Taking R_1 and R_2 to be the same space R , the sum

$$S(x, y) = K_1(x, y) + K_2(x, y)$$

is clearly a p.m. rel. R and the tilde correspondence yields.

Theorem 1. $S(x, y)$ is the r.k. of the space G of all functions on E to R which have a decomposition $g(x) = f_1(x) + f_2(x)$, for some $f_1 \in F_1$ and $f_2 \in F_2$, with the norm given by

$$\|g\|^2 = \min \{ \|f_1\|^2 + \|f_2\|^2 \},$$

the minimum being taken over all such decompositions of $g(x)$.

In the general case, $R_1 \neq R_2$, we can define a formal sum, the direct sum discussed below, even though R_1 and R_2 are unrelated. However, if R_1 and R_2 are appropriately related we can extend the above theorem :

Theorem 2. If there exist continuous linear transformations σ_1 and σ_2 mapping R_1 and R_2 into the same space R then

$$S(x, y) = \sigma_1 K_1(x, y) \sigma_1^* + \sigma_2 K_2(x, y) \sigma_2^*$$

is a p.m. rel. R and is the r.k. of the space G of all functions $g(x)$ on E to R of the form $g(x) = \sigma_1 f_1(x) + \sigma_2 f_2(x)$, for some $f_1 \in F_1$ and $f_2 \in F_2$, under the norm

$$\|g\|^2 = \min \left\{ \inf_{T_{\sigma_1}(f'_1 - f_1)=0} \|f'_1\|^2 + \inf_{T_{\sigma_2}(f'_2 - f_2)=0} \|f'_2\|^2 \right\},$$

the minimum being taken over all such decompositions of $g(x)$.

This is an obvious consequence of Theorem 1 and Theorem 4 of §3.

A few special cases are the following:

1) If R_1 is a closed subspace of R_2 then the sum of K_2 and the natural extension of K_1 to R_2 is the r.k. of the space of all sums $g(x) = f_1(x) + f_2(x)$, $f_1 \in F_1$ and $f_2 \in F_2$, with the norm

$$\|g\|^2 = \min_{g(x)=f_1(x)+f_2(x)} \{ \|f_1\|^2 + \|f_2\|^2 \}.$$

2) If R_2 is the space R_1 under a weaker topology we conclude that the sum of K_1 and its topological restriction K_2 is the r.k. of the space of all functions on E to R_2 which have the form $g(x) = f_1(x) + f_2(x)$ as above.

If we take the space R in the theorem to be the direct sum $R_1 \dot{+} R_2$ and σ_1 and σ_2 to be the injection mappings of R_1 and R_2 into R then σ_1^* and σ_2^* are respectively the projections carrying an element of $R_1^* \dot{+} R_2^*$ into its first and second coordinates, i.e., $\sigma_1^*(r_1^* + r_2^*) = r_1^*$ and $\sigma_2^*(r_1^* + r_2^*) = r_2^*$. The theorem then yields a sum, the direct sum of K_1 and K_2 :

$$[K_1(x, y) \dot{+} K_2(x, y)](r_1^* + r_2^*) = K_1(x, y)r_1^* + K_2(x, y)r_2^*,$$

as a p.m. rel. $R_1 \dot{+} R_2$. The space of which $K_1 \dot{+} K_2$ is the r.k. is clearly the (orthogonal) direct sum of F_1 and F_2 with the norm

$$\|f_1 + f_2\|^2 = \|f_1\|^2 + \|f_2\|^2.$$

Differences. For two p.m.'s K and K' relative to the same space R we write $K \ll K'$ whenever $K'(x, y) - K(x, y)$ is a p.m.rel. R . It is easily seen that \ll is a partial ordering of the set of p.m.'s rel. R .

The tilde correspondence gives

Theorem 3. If $K \ll K'$ and F, F' and $\| \cdot \|, \| \cdot \|'$ are the corresponding spaces and norms then $F \subset F'$ and $\|f\| \geq \|f\|'$ for all $f \in F$.

Theorem 4. If K' is the r.k. of F' , norm $\| \cdot \|'$, and if $F \subset F'$ is a Hilbert space under the norm $\| \cdot \|$ such that for every $f \in F$, $\|f\| \geq \|f\|'$, then F possesses a r.k. K satisfying $K \ll K'$.

As in the case of the sums, we can apply these theorems to the situation where the range spaces are different, say R_1 and R_2 provided there exist continuous linear transformations σ_1 and σ_2 of R_1 and R_2 into a space R . Thus, in particular, the theorems can be applied under certain circumstances to natural extensions, quotients and topological restrictions of given r.k.'s. The results obtained may be useful as computational devices but since their general interest is limited we restrict ourselves to the samples already given.

Finally, the construction of the space corresponding to the difference in Theorem 4 and the correspondence between decompositions $K = K_1 + K_2$ of a r.k. and decompositions $I = L_1 + L_2$ of the identity operator in F into positive operators can be carried over verbatim to the present case by the tilde correspondence.

§7. Tensor product.

For the pair of proper functional spaces F_1 and F_2 with basic sets E_1 and E_2 and range spaces R_1 and R_2 we form the tensor product of their kernels as a transformation of $R_1^* \otimes R_2^*$ into $R_1 \otimes R_2$ and denote it by K ,

$$K(x_1, x_2; y_1, y_2) = K_1(x_1, y_1) \otimes K_2(x_2, y_2) .$$

Applying the tilde correspondence to the spaces F_1 and F_2 and their kernels and computing the product of \tilde{K}_1 and \tilde{K}_2 we find

$$\begin{aligned} & \tilde{K}_1(x_1, r_1^*; y_1, s_1^*) \cdot \tilde{K}_2(x_2, r_2^*; y_2, s_2^*) \\ &= \overline{r_1^* \otimes r_2^*, K(x_1, x_2; y_1, y_2) s_1^* \otimes s_2^*} , \end{aligned}$$

that is, the restriction to the elementary products of \tilde{K} . Denote this restriction by \tilde{K}_0 .

As the product of two ordinary r.k.'s \tilde{K}_0 is an ordinary r.k. in the set $E_1 \times E_2 \times R_1^* \times R_2^*$. It is, according to the usual product theorem, the r.k. of the tensor product $\tilde{F}_1 \otimes_h \tilde{F}_2$ considered as a space of functions on $E_1 \times E_2 \times R_1^* \times R_2^*$. This space is the functional completion of $\tilde{F}_0 = \tilde{F}_1 \otimes \tilde{F}_2$ of all functions of the form

$$\tilde{f}(x_1, x_2; r_1^* \otimes r_2^*) = \sum_{k=1}^n \tilde{f}_1^{(k)}(x_1, r_1^*) \cdot \tilde{f}_2^{(k)}(x_2, r_2^*) ,$$

with $\tilde{f}_1^{(k)} \in \tilde{F}_1$ and $\tilde{f}_2^{(k)} \in \tilde{F}_2$, under the norm given by the scalar product

$$(\tilde{f}, \tilde{g}) = \sum_{k=1}^n \sum_{l=1}^m (\tilde{f}_1^{(k)}, \tilde{g}_1^{(l)}) (\tilde{f}_2^{(k)}, \tilde{g}_2^{(l)}) .$$

The characterization of the elements of the complete space in terms

of complete orthonormal systems in the two factor spaces could be carried over from the usual theory but it will be sufficient for us to deal with the space \tilde{F}_0 .

The elements of \tilde{F}_0 have unique antilinear extensions to $R_1^* \otimes R_2^*$ and the scalar product can be transferred to these extensions, giving a space of functions on $E_1 \times E_2 \times (R_1^* \otimes R_2^*)$ which we denote by \tilde{F}_1 .

It is clear that \tilde{F}_1 corresponds to the space $F_1 \otimes F_2$, considered as a space of functions in $E_1 \times E_2$ to $R_1 \otimes R_2$ by means of the evaluations

$$T_{x_1 x_2} = T_{x_1}^{(1)} \otimes T_{x_2}^{(2)},$$

where $T_{x_1}^{(1)}$ and $T_{x_2}^{(2)}$ are the evaluations of F_1 and F_2 , by means of the tilde correspondence.

The function \tilde{K}_0 can be similarly extended to a function \tilde{K} on $E_1 \times E_2 \times R_1^* \otimes R_2^*$. \tilde{K} belongs to \tilde{F}_1 for each fixed choice of its second argument and has the reproducing property in \tilde{F}_1 since \tilde{K}_0 has the corresponding property in \tilde{F}_0 (note that \tilde{F}_0 being incomplete we don't speak of a r.k. for \tilde{F}_0).

We may now construct the topology $\tilde{\tau}$ on $R_1^* \otimes R_2^*$ by means of \tilde{F}_1 . This topology may fail to be Hausdorff but if we strengthen it by adding all the neighborhoods of zero of the form, all $r^* \in R_1^* \otimes R_2^*$ such that

$$| \langle r^*, r_1 \otimes r_2 \rangle | < \epsilon,$$

for any $\epsilon > 0$ and any $r_1 \in R_1$, $r_2 \in R_2$, we obtain a topology τ' on $R_1^* \otimes R_2^*$ which is stronger than $w(R_1^* \otimes R_2^*, R_1 \otimes R_2)$ (and hence is a Hausdorff topology).

The conjugate of $R_1^* \otimes R_2^*$ with topology τ' is the space $R_1 \otimes_{\tau'} R_2$. By the general theorem on functional completion, since a functional completion exists for \tilde{F}_1 such a completion exists also for $F_1 \otimes F_2$ giving $F_1 \otimes_h F_2$ with range space $R_1 \otimes_{\tau'} R_2$. Hence,

Theorem 1. The tensor product K of K_1 and K_2 is a p.m. rel. $R_1 \otimes_{\tau'} R_2$ and is the r.k. of $F_1 \otimes_h F_2$ with this range space.

Finally if $E_1 = E_2 = E$, applying the restriction theorem to the diagonal set in $E_1 \times E_2$ we obtain

Theorem 2. The tensor product

$$K(x, y) = K_1(x, y) \otimes K_2(x, y)$$

of two positive matrices defined in E is a p.m. and is the r.k. of the space of all restrictions to the diagonal set in $E \times E$ of functions in $F_1 \otimes_h F_2$ with range space $R_1 \otimes_{\tau'} R_2$, under the norm

$$\|f'\| = \min \|f\| ,$$

the minimum being taken over all $f \in F_1 \otimes_h F_2$ whose restriction to the diagonal is f' .

§8. Limits of reproducing kernels.

The theorems proved in the case of ordinary kernels cover the two cases of a decreasing sequence of kernels corresponding to a decreasing sequence of classes of functions defined in an increasing sequence of sets and the reverse situation. Without any essential alteration of the proofs, these theorems may be extended to the case where the basic sets E_n are the members of, respectively, an inductive or a projective system. We shall need the results in this form so we begin by stating them briefly for later reference.

Case A. We suppose that $\{E_n, \pi_n\}$ is an inductive system of sets and denote by x_n the generic element of E_n and by E the inductive limit.

We assume either of the following equivalent conditions:

1°. F_n is a p.f. Hilbert space on E_n for each n such that the sequence of classes F_n is decreasing in the sense

$$f_{n+1}(\pi_n x_n) \in F_n, \quad \underline{\text{for all}} \quad f_{n+1} \in F_{n+1} \quad \underline{\text{and all}} \quad n,$$

and the norms satisfy

$$\|f_{n+1}(\pi_n x_n)\|_n \leq \|f_{n+1}\|_{n+1}, \quad \underline{\text{for all}} \quad n.$$

2°. F_n is the space whose r.k. is $K_n(x_n, y_n)$ and these r.k.'s form a decreasing sequence in the sense that

$$K_{n+1}(\pi_n x_n, \pi_n y_n) \ll K_n(x_n, y_n)$$

as p. matrices on E_n for each n .

Define a space F_0 of functions on E as follows. Let $f_0(\{x_n\}) \in F_0$ if and only if

$$f_0(\{x_n\}) = I\text{-}\lim_n f_n(x_n)$$

for some inductive sequence of functions $\{f_n\}$ with the properties that $f_n(x_n) \in F_n$ for all n and $\lim_n \|f_n(x_n)\|_n < \infty$ and put

$$\|f_0\|_0 = \lim_n \|f_n\|_n.$$

Theorem 1. The limit of the kernels K_n ,

$$K_0(\{x_n\}, \{y_n\}) = \lim_n K_n(x_n, y_n)$$

is a p.m. on E and is the r.k. of the space F_0 .

Case B. We suppose that $\{E_n, \pi_n\}$ is a projective system and that E denotes the projective limit.

Our assumption is either of the equivalent conditions :

1°. F_n is a p.f. Hilbert space on E_n for each n such that the classes increase in the sense

$$f_n(\pi_n x_{n+1}) \in F_{n+1}, \text{ for all } f_n \in F_n \text{ and all } n;$$

and the norms satisfy

$$\|f_n(\pi_n x_{n+1})\|_{n+1} \leq \|f_n\|_n, \text{ for all } n.$$

2°. F_n is the space whose r.k. is $K_n(x_n, y_n)$ and the kernels form an increasing sequence in the sense

$$K_n(\pi_n x_{n+1}, \pi_n y_{n+1}) \ll K_{n+1}(x_{n+1}, y_{n+1}) \text{ for all } n.$$

The sequence of positive numbers $K_n(y_n, y_n)$ for any $\{y_n\} \in E$ is an increasing sequence under our assumptions. We denote by E_0 the subset of E of all $\{y_n\}$ such that $\lim_n K_n(y_n, y_n) < \infty$ and suppose that E_0 is not empty.

Consider the class F'_0 of all functions $f_0(\{x_n\})$ defined on E_0 which are projective limits on E_0 of projective sequences of functions $\{f_n\}$ with $f_n \in F_n$ for each n . For such a sequence $\{f_n\}$ the assumption 1° assures that $\|f_n\|_n$ is a decreasing sequence for sufficiently large n . We norm F'_0 by putting

$$\|f_0(\{x_n\})\|'_0 = \inf_{\{f_n\}} \lim_n \|f_n\|_n ,$$

the infimum being over all projective sequences $\{f_n\}$ giving f_0 .

If F'_0 has no functional completion in this norm we can decrease the norm so that the completion exists. Namely, putting

$$\|f_0(\{x_n\})\|_0 = \inf_{\{f_0^{(k)}\}} \|f_0^{(k)}(\{x_n\})\|'_0 ,$$

the infimum being over all Cauchy sequences $\{f_0^{(k)}\}$ in F'_0 converging to f_0 . With this norm the completion exists. We denote it by F_0 .

Theorem 2. The limit of the kernels $K_n(x_n, y_n)$

$$K_0(\{x_n\}, \{y_n\}) = \lim_n K_n(x_n, y_n)$$

is a p.m. on E_0 and is the r.k. of F_0 .

We apply these theorems to our case by means of the tilde correspondence, taking first the analogue of case A.

Suppose $\{E_n, \pi_n\}$ is an inductive system, E its inductive limit and suppose that $\{R_n, \sigma_n\}$ is a projective system of l.c. spaces, R the projective limit.

We assume one of the equivalent conditions:

1° . For each n , F_n is a p. f. Hilbert space rel. R_n , such that the sequence F_n is decreasing in the sense that

$$\sigma_n f_{n+1}(\pi_n x_n) \in F_n, \quad \text{for all } f_{n+1} \in F_{n+1} \text{ and all } n,$$

and the norms increase, i.e.

$$\|\sigma_n f_{n+1}(\pi_n x_n)\|_n \leq \|f_{n+1}\|_{n+1}.$$

2°. F_n is the space with r.k. $K_n(x_n, y_n)$ and the kernels form a decreasing sequence in the sense

$$\sigma_n K_{n+1}(\pi_n x_n, \pi_n y_n) \sigma_n^* \ll K_n(x_n, y_n), \quad \text{all } n.$$

Under these circumstances $\{E_n \times R_n^*, \pi_n \times \sigma_n^*\}$ is an inductive system and the hypotheses of Theorem 1 are satisfied by \tilde{F}_n , or by \tilde{K}_n in case 2° is taken. Consequently,

$$\tilde{K}_0(\{x_n\}, \{r_n^*\}; \{y_n\}, \{s_n^*\}) = \lim \tilde{K}_n(x_n, r_n^*; y_n, s_n^*)$$

exists and is a p.m. on $E \times R^*$, where R^* is the inductive limit of $\{R_n^*, \sigma_n^*\}$. \tilde{K}_0 is the r.k. of the space of all \tilde{f}_0 such that

$$\tilde{f}_0(\{x_n\}, \{r_n^*\}) = I\text{-}\lim_n \tilde{f}_n(x_n, r_n^*)$$

for some inductive sequence \tilde{f}_n as described in Theorem 1.

We wish now to apply the tilde correspondence to the space \tilde{F}_0 and obtain a proper functional space with $R = P\text{-}\lim \{R_n, \sigma_n\}$ as range space. For this we must show that all elements of the form $\tilde{f}_0(\{x_n\}, \{r_n^*\})$ in $(R^*)'$ belong to R . Since, for an inductive sequence $\tilde{f}_n(x_n, r_n^*)$ giving the element $\tilde{f}_0(\{x_n\}, \{r_n^*\})$ we have

$$\tilde{f}_n(x_n, r_n^*) = f_n(x_n) \in R_n$$

we must show that $\{f_n(x_n)\} \in R$ and that for any $r^* = \{r_n^*\} \in R^*$

$$\langle r^*, \{f_n(x_n)\} \rangle = \tilde{f}_0(\{x_n\}, r^*).$$

By the inductive property of the \tilde{f}_n , we have

$$\tilde{f}_n(x_n, r_n^*) = \tilde{f}_{n+1}(\pi_n x_n, \sigma_n^* r_n^*).$$

Hence,

$$\langle r_n^*, f_n(x) \rangle = \langle \sigma_n^* r_n^*, f_{n+1}(\pi_n x_n) \rangle = \langle r_n^*, \sigma_n f_{n+1}(x_{n+1}) \rangle,$$

since $\{x_n\} \in E$, and therefore

$$f_n(x_n) = \sigma_n f_{n+1}(x_{n+1}),$$

proving the first point. Further, the definition of $\langle r^*, r \rangle$ for $r \in R$ and $r^* \in R^*$ together with the definition of $I\text{-}\lim \tilde{f}_n(x_n, r_n^*)$ show that the required equation is satisfied.

The fact that the topology on R^* is not necessarily Hausdorff does not affect these considerations since we may consider elements of R^* which cannot be distinguished by elements of R as giving the same functional on R .

The transformation $K_0(\{x_n\}, \{y_n\})$ of R^* into R corresponding to \tilde{K}_0 is well defined, that is, has the same value for indistinguishable elements of R^* , because of the manner in which it is obtained. Moreover, K_0 is a p.m. relative to the l.c.s. R and is the r.k. of the space F_0 of functions on E to R defined from \tilde{F}_0 by the tilde correspondence.

It remains for us to interpret K_0 as the limit in an appropriate sense of the kernels K_n . For $r^* = \{r_k^*\} \in R^*$ and N sufficiently large, define a sequence $\{r_{p,N}\}$ of elements $r_{p,N} \in R_p$, $p = 1, 2, \dots$, by

$$r_{p,N} = \begin{cases} K_p(x_p, y_p) r_p^*, & p \geq N \\ \sigma_p r_{p+1,N}, & 1 \leq p \leq N-1 \end{cases}$$

Intuitively, the sequences $\{r_{p,N}\}$ have initial segments satisfying the requirement for a sequence to belong to R . Considered as elements of $\prod R_n$ they converge there in the topology $w(\prod R_n, \sum R_n^*)$ to an element of R , as $N \rightarrow \infty$. Choosing this value for $K_0(\{x_n\}, \{y_n\})r^*$, we see that it agrees with the value obtained by the tilde correspondence. Consequently, it is in this sense that we are to understand the expression $\lim_n K_n$ in the

Theorem 3. The limit

$$K_0 = \lim_n K_n$$

is a p.m. rel. R on the basic set E and is the r.k. of F_0 .

Some special cases of the setting for this theorem are of interest. Notably the case where the spaces R_n form a decreasing sequence of closed subspaces of a space R_0 and the transformations σ_n are the injections. If a sequence of kernels is defined having the properties of the hypothesis 2° then the theorem gives as the limit a p.m. relative to the intersection of the spaces R_n . A second special case is that in which each R_n is a quotient space of the succeeding space R_{n+1} and σ_n is the quotient map.

We turn now to the case B, and let $\{E_n, \pi_n\}$ be a projective system with limit E and $\{R_n, \sigma_n\}$ be an inductive system with limit R . We phrase the equivalent hypotheses 1° and 2° as follows, in this case.

1°. For each n , F_n is a p.f. Hilbert space rel. R_n such that $\{F_n\}$ is an increasing sequence in the sense that for each n

$$\sigma_n f_n(\pi_n x_{n+1}) \in F_{n+1}, \quad \underline{\text{for all}} \quad f_n \in F_n,$$

and the norms decrease, i.e.,

$$\|\sigma_n f_n(\pi_n x_{n+1})\|_{n+1} \leq \|f_n\|_n, \text{ all } n.$$

2°. E_n is the space with r.k. $K_n(x_n, y_n)$ and for every n

$$\sigma_n K_n(\pi_n x_{n+1}, \pi_n y_{n+1}) \sigma_n^* \ll K_{n+1}(x_{n+1}, y_{n+1}).$$

$\{E_n \times R_n^*, \pi_n \times \sigma_n^*\}$ is a projective system in this case and has P-limit $E \times R^*$, where $R^* = P\text{-lim} \{R_n^*, \sigma_n^*\}$. The \tilde{K}_n can therefore be examined by means of Theorem 2.

The space R may not be Hausdorff. Let R' denote the quotient space of R modulo its zero subspace and q denote the quotient map.

We remark first that in this case the r.k. $K_n(x_n, y_n)$ determines a transformation of R^* into R' . Namely,

$$K_{0n}(\{x_i\}, \{y_i\}) \{r_i^*\} = q \sigma_n \circ K_n(\pi_n \circ \{x_i\}, \pi_n \circ \{y_i\}) \sigma_n^* \{r_i^*\}.$$

It is easy to see that

$$\tilde{K}_{0n}(\{x_i\}, \{r_i^*\}; \{y_i\}, \{s_i^*\}) = \tilde{K}_n(x_n, r_n^*; y_n, s_n^*)$$

with the understanding that $\langle r^*, r' \rangle$, for $r^* \in R^*$ and $r' \in R'$, has the value $\langle r^*, r \rangle$ for any $r \in R$ such that $r' = qr$.

The convergence of the kernels K_n is to be taken in the sense that the kernels K_{0n} converge weakly as transformations of R^* into R' and we see from the above that this is equivalent to the convergence of the \tilde{K}_n .

Denote by E_0 the maximal subset of E such that for every $\{r_i^*\} \in R^*$

$$\lim \tilde{K}_{0n}(\{y_i\}, \{r_i^*\}; \{y_i\}, \{r_i^*\}) < \infty$$

when $\{y_i\} \in E_0$. Then E_0 is contained in the subset of $E \times R^*$ which must be chosen for the application of Theorem 2. to the spaces \tilde{F}_n and their kernels \tilde{K}_n . The space \tilde{F}_0 obtained in applying Theorem 2 is a p.f.s. with

the limit of \tilde{K}_{0n} as r.k. and hence we may apply the restriction theorem to obtain a space \tilde{F}'_c of functions on $E_0 \times R^*$ whose r.k. is the restriction there of $\lim \tilde{K}_{0n}$.

\tilde{F}'_c is the functional completion of the space of all restrictions of functions of the type

$$\tilde{f}_0(\{x_n\}, \{r_i^*\}) = P\text{-}\lim \tilde{f}_n(x_n, r_n^*)$$

where $\{\tilde{f}_n\}$ is some projective sequence with $\tilde{f}_n \in \tilde{F}'_n$. Denote this space by \tilde{F}'_c . As in the previous theorem we wish to show that each functional of the form $\tilde{f}_0(\{x_i\}, \{r_i^*\})$ on R^* is an element of R' . Choosing any projective sequence \tilde{f}_n giving \tilde{f}_0 , put $r = \{f_n(x_n)\}$, $n \geq n_0$ for a suitable n_0 . Then

$$\tilde{f}_{n+1}(x_{n+1}, r_{n+1}^*) = \tilde{f}_n(\pi_n x_{n+1}, \sigma_n^* r_{n+1}^*) ,$$

$$\langle r_{n+1}^*, f_{n+1}(x_{n+1}) \rangle = \langle \sigma_n^* r_{n+1}^*, f_n(\pi_n x_{n+1}) \rangle = \langle r_{n+1}^*, \sigma_n f_n(x_n) \rangle .$$

Thus $f_{n+1}(x_{n+1}) = \sigma_n f_n(x_n)$, that is, $r \in R$ and it is clear that $\langle \{r_i^*\}, r \rangle = \tilde{f}_0(\{x_i\}, \{r_i^*\})$. We may therefore choose qr as the element $\tilde{f}_0(\{x_i\}, \{r_i^*\})$ of R' .

Finally, we construct the $\tilde{\tau}$ topology on R^* determined by the space \tilde{F}'_c and adjoint to its neighborhoods the $w(R^*, R')$ neighborhoods obtaining a stronger topology τ' . Applying the theory of Section 5, we obtain a functional completion F'_c of \tilde{F}'_c having as range space the completion R_c of R' determined by the topology τ' .

We may now state

Theorem 4. The limit $K_0(\{x_i\}, \{y_i\}) = \lim_n K_n(x_n, y_n)$

is a p.m. rel. R_c on E_0 and is the r.k. of F'_c .

The special cases of this situation which occur naturally are first, the case of an increasing sequence R_n of subspaces of a fixed space, with σ_n as the injection, and secondly, the case where R_n forms a chain of successive quotient spaces $R_{n+1} = R_n/N_n$ with the quotient map for σ_n .

It was in order to obtain general theorems which include the interesting special cases, such as those cited after Theorem 3 and the above, that it was necessary to consider the limits in the setting of inductive and projective systems.

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